Homogenous and Non-Homogenous Linear Partial Differential Equations of Second & Third Order with Constant Coefficients

by

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Linear Partial Differential Equation → A partial differential Equation is an equation which contains partial derivatives of any order but non of the partial derivatives appears in degree higher than unity (one) is called a linear partial differential Equation.

Remarks (1) → If the coefficients of each differential coefficient is Constant, then it is called Linear Partial Differential Equation with Constant Co-efficients

Remarks (2) → If all the partial derivatives are of the same order, then the linear partial differential Equation is said to be **Homogeneous Linear Partial Differential Equation**

Example-(1) → $a\frac{\partial^2 z}{\partial x^2} + 2h\frac{\partial^2 z}{\partial x \partial y} + b\frac{\partial^2 z}{\partial y^2} = f(x,y)$

where a, b & h are constants & is called a homogeneous linear partial Equation with constant coefficient of order 2.

Note-(1) → When all the partial derivatives involve in Equation are not of the same order, then it is called **Non-Homogeneous linear partial Equation with Constant Co-efficients**.

Examples-(1) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = \sin(2x+3y)$

(2) $3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} - 5\frac{\partial z}{\partial y} = e^{x+4y}$

(1) & (2) are the Examples of Non-Homogeneous linear differential Equations of order 2 with constant Co-efficients.

Notations Used

Here we denote (1) $\frac{\partial}{\partial x}$ by D or D_x

(2) $\frac{\partial}{\partial y}$ by D' or D_y

(3) $\frac{\partial^2}{\partial x^2}$ by D^2 or D_x^2

(4) $\frac{\partial^2}{\partial y^2}$ by D'^2 or D_y^2

(5) $\frac{\partial^2}{\partial x \partial y}$ by DD' or $D_x D_y$

(6) $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$,

$\frac{\partial^2 z}{\partial y^2} = t$.

$x+2y$

Methods to solve Linear Partial Differential Equations with Constant Coefficients

Firstly, we will discuss the Homogenous Linear Partial Differential Equations with Constant Coefficients.

i.e If the Equation is of the type $F(D, D')Z = F(x, y)$

or

$$F(D_x, D_y)Z = F(x, y).$$

We know that solution of Ordinary linear differential Equation consists of two parts, the Complementary function (C.F.) and the particular Integral (P.I.)

Note-1 → C.F Contains as many arbitrary constants as order of differential Equation and P.I does not contain any arbitrary constants. Hence the general solution will contain as many arbitrary constants as order of partial Differential Equation.

(2) If R.H.S $= F(x, y) = 0$, then P.I (Particular Integral) $= 0$

∴ solution of eqn is $\boxed{y = C.F}$

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ILLUSTRATIVE EXAMPLES

Q1 → Solve $(D^2 - 3DD' + 2D'^2)Z = 0$

Sol: Given Partial Differential Equation is

$$(D^2 - 3DD' + 2D'^2)Z = 0 \quad \text{---(1)}$$

Auxilliary Equation is

$m^2 - 3m + 2 = 0$ (by taking $D = m$, $D' = 1$ & $Z = 1$)

or $m^2 - m - 2m + 2 = 0$

or $m(m-1) - 2(m-1) = 0$

or $(m-1)(m-2) = 0$

or $m = 1, 2$ (real & distinct)

∴ general sol. of Equation (1) is

$$\boxed{Z = \phi_1(y + x) + \phi_2(y + 2x)} \quad \text{Ans}$$

Q2. Solve $(D^2 - 4DD' + 4D'^2)Z = 0$

Sol: We have $(D^2 - 4DD' + 4D'^2)Z = 0$ ---(1)

A.E is $m^2 - 4m + 4 = 0$ [∵ $D = m$, $D' = 1$, $Z = 1$]

or $(m-2)^2 = 0$

or $(m-2)(m-2) = 0$

i.e $m = 2, 2$. (real & repeated)

∴ general sol. of (1) is

$$\boxed{Z = \phi_1(y + 2x) + x\,\phi_2(y + 2x)} \quad \text{Ans}$$

Q3:→ Solve $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x^2 \partial y} + 12\frac{\partial^3 z}{\partial x \partial y^2} = 0$

Solution: We have, in symbolic form, $(D^3 - 7D^2D' + 12DD'^2)z = 0$ —— (1)

$\therefore$ A.E is $m^3 - 7m^2 + 12m = 0$ —— $\because D = m, D' = 1$ & $z = 1$.

or $m(m^2 - 7m + 12) = 0$.

or $m = 0, 3, 4$.

$\therefore$ general solution of (1) is

$$\boxed{z = \phi_1(y + 0x) + \phi_2(y + 3x) + \phi_3(y + 4x)}$$ Ans

Q4: Solve $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$.

Sol: Here $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$ —— (1)

A.E is $m^3 - 6m^2 + 11m - 6 = 0$ —— (*) (by putting $D = m, D' = 1$).

$m = 1$ satisfies the Equation (*)

$\therefore m = 1$ is one root of (*).

Now, $(m - 1)$ is one factor.

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$\therefore (m^2 - 5m + 6)$ is other factor.

$\therefore$ from (*) $(m - 1)(m^2 - 5m + 6) = 0$.

$\therefore m = 1, 2, 3$.

$\therefore$ general solution is

$$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$$

Solve 5 → $\frac{\partial^3 z}{\partial x^3} - 6\frac{\partial^3 z}{\partial x^2 \partial y} + 12\frac{\partial^3 z}{\partial x \partial y^2} - 8\frac{\partial^3 z}{\partial y^3} = 0$

Solution: In symbolic form, we have

$$(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$$

Its corresponding A.E is

$$m^3 - 6m^2 + 12m - 8 = 0 \quad (\text{by putting } D = m, D' = 1)$$

$\therefore m = 2, 2, 2$

$\therefore$ The general sol. is

$$\boxed{z = \phi_1(y+2x) + x\phi_2(y+2x) + x^2\phi_3(y+2x)}$$

Q6:- Solve $(D^2 + a^2D'^2)z = 0$

Solution: we have in symbolic form $(D^2 + a^2D'^2)z = 0$

$\therefore$ A.E is $m^2 + a^2 = 0$

i.e $m^2 = -a^2$

i.e $\boxed{m = \pm ia}$

$\therefore$ The general sol. is

$$z = \phi_1(y + iax) + \phi_2(y + (-ia)x)$$

$$\boxed{\text{or } z = \phi_1(y + iax) + \phi_2(y - iax)}$$

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